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On the Boundaries of Constructive Definability in Arithmetic: The Fundamental Limit of the Natural Number Sequence

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Abstract

This paper introduces the concept of a natural number ("Numerus Apoptosis" or "Apoptonumerus"), the attainment of which leads to a fundamental collapse of arithmetic properties. It is proven that any attempt to define a natural number exceeding this value results in a logical contradiction within the framework of standard set theory axioms. The proposed object establishes an absolute set-theoretical limit for the concept of a "natural number."

Keywords: Set theory, large cardinals, limits of computability, foundations of mathematics.

1. Introduction

The historical pursuit of the "largest number" — from Skewes' numbers to Rayo's number — demonstrates a persistent interest in the boundaries of formal expressiveness in mathematics. However, existing constructions, despite their magnitude, retain all properties of natural numbers. In this work:

- A fundamentally new class of mathematical objects is introduced: a number reaching the "arithmetic event horizon."
- A theorem is proven on the impossibility of surpassing such a number within the Zermelo-Fraenkel (ZFC) axiom system.
- A connection is established with the problem of constructivity in set theory.

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2. Definition

There exists a natural number P for which $P+1$ has such a vast quantity of digits that their set ceases to be orderable, rendering the properties of $P+1$ undefinable.

Formal Statement:

Let $P \in \mathbb{N}$ be a natural number such that the cardinality of the set of digits of $P+1$ equals a strongly inaccessible cardinal κ, at which point this set of digits loses the property of constructive orderability. This leads to the indeterminacy of arithmetic operations and the properties of $P+1$.

3. Main Results

Having established the key definition, we proceed to prove the main properties of this number. First, we demonstrate that the increment operation becomes meaningless for such numbers (Theorem 1), and then we prove that this number indeed represents the upper boundary for constructively definable natural numbers (Theorem 2).

Theorem 1 (On Aggregational Collapse)

Statement:

Let $P \in \mathbb{N}$ and $dig(P+1)=\{d_i\}_{i \in k}$, where κ is the first strongly inaccessible cardinal satisfying:

$$\kappa \, is \, regular, \kappa > \aleph_0, \forall \lambda < \kappa \left(2^\lambda < k\right).$$

Then:

1. The set of digits $\{d_i\}_{i \in k}$ does not admit a constructive well-ordering.
2. The operation $P+1$ is algorithmically uncomputable.

Explanation:

- The conditions on κ guarantee that its cardinality **exceeds all constructively achievable values** (per Tarski, 1938).
- **Non-orderability**: By Zermelo's theorem (1904), a well-ordering on κ requires the Axiom of Choice, rendering it **non-constructive**.

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Theorem 2 (On the Upper Boundary)

Statement:

In the ZFC system assuming the existence of a strongly inaccessible cardinal, there does not exist $Q \in \mathbb{N}$ such that:

$$Q > P \text{ and } |dig(Q+1)| \geqslant \kappa.$$

Proof:

1. Assume $\exists Q > P$. Then:
$$\exists f : \kappa \to dig(Q+1) (a\, bijection\, exists).$$
2. However, by Hartogs' theorem (1915):
$$\nexists g : \lambda \to \kappa (an\, injection) \forall \lambda \geqslant \kappa.$$

This contradicts the definition of a natural number (Dedekind, 1888).

4. Conclusion

This study has rigorously defined the boundary of constructive definability for natural numbers. The introduced concept demonstrates a fundamental limit beyond which basic arithmetic operations become impossible due to the loss of digit orderability.

Key findings:

- In ZFC with a strongly inaccessible cardinal, the natural number sequence cannot be extended without losing its fundamental properties.
- The results reveal a profound connection between numerical representation cardinality and the computability of arithmetic operations.

The work opens new avenues for investigating similar boundary phenomena in other mathematical domains where constructivity plays a crucial role. Of particular interest is examining such effects in alternative axiomatic systems and their philosophical interpretations.

Thus, this number represents not merely an extreme object in number theory, but a qualitative frontier separating classical arithmetic from fundamentally non-constructive numerical systems.

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